Applecross Senior High School

Name : SOLUTIONS.

Mathematics Method 3 Test 1, 2017

Time Allowed : 55 minutes.

Note : For both Section 1 and section 2, working out must be shown for full marks to be awarded.

Section 1 : [/ 17 marks] Section 2 : [/ 31 marks] Total : [/ 48 marks] = ____ %

Section 1 : Calculator and Resource Free Time : 20 minutes

1. [3,2 = 5 marks]

Differentiate the following with respect to x.

a) $f(x) = \frac{-x}{x^2+1}$ { Express numerator in simplest form }

$$f'(x) = \frac{-(x^2+1) - (-x)(2x)}{(x^2+1)^2}$$ ✓✓ ($-\frac{1}{2}$ mark per error / missing)

$$= \frac{-x^2-1+2x^2}{(x^2+1)^2}$$

$$= \frac{x^2-1}{(x^2+1)^2}$$ ✓

b) $y = (1-x)^3 (1 + \frac{2}{x})^2$ { Apply the product rule but do not simplify } ✓

$$\frac{dy}{dx} = \left[3(1-x)^2(-1)(1+\tfrac{2}{x})^2\right] + \left[2(1+\tfrac{2}{x})(-\tfrac{2}{x^2})(1-x)^3\right]$$ ✓

$$-3(1-x)^2(1+\tfrac{2}{x})^2 \qquad -\frac{4}{x^2}(1+\tfrac{2}{x})(1-x)^3$$

2. [2,2= 4 marks]

A particle moves in a straight line such that its velocity, v m/s, depends upon displacement , x m , from some fixed point O according to the rule v = 5x-4

a) Find an expression in terms of x for the acceleration of the particle.

$$a = \frac{dv}{dt}, \quad V = \frac{dx}{dt}, \quad \frac{dv}{dx} = 5$$

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$= 5(5x-4)$$

$$= \underline{25x - 20.}$$

✓ only if $5ms^{-2}$

b) Determine the displacement and the acceleration of the particle when v = 6m/s.

$$5x - 4 = 6$$

$$x = 2$$

when $x = 2$ ✓

EL

2m.

$$a = 25x - 20$$

$$= 25(2) - 20$$

$$\underline{a = 30 m/s^2}$$ ✓

$a = 5ms^{-2}$ ✓

3. [8 marks]

The equation of the tangent to the curve $y = ax^3 - bx^2 + 2$ where $x = -1$ is $y = 18x + c$.

The curve has a point of inflection when $x = 1$.

Find the values of a, b and c.

[Note : Working out must be shown]

$y = ax^3 - bx^2 + 2$

$\frac{dy}{dx} = 3ax^2 - 2bx$ ✓

Tangent is $y = 18x + c$ when $x = -1$

$\therefore \frac{dy}{dx} = 18$ when $x = -1$. ✓

when $x = -1$, $\frac{dy}{dx} = 18$

$3ax^2 - 2bx = 18$

$3a(-1)^2 - 2b(-1) = 18$

$3a + 2b = 18$ —— (1) ✓

Point of Inflection when $\frac{d^2y}{dx^2} = 0$.

$\frac{d^2y}{dx^2} = 6ax - 2b$.

when $x = 1$, $\frac{d^2y}{dx^2} = 0$.

$6ax - 2b = 0$.

$6a(1) - 2b = 0$. ✓

$6a - 2b = 0$ —— (2).

Solve Simultaneously

$3a + 2b = 18$ —— (1)

$6a - 2b = 0$ —— (2)

(1) + (2) $9a = 18$ ✓

$a = 2$

$6a - 2b = 0$

$12 - 2b = 0$

$b = 6$ ✓

when $x = -1$

$y = 2x^3 - 6x^2 + 2$.

$y = 2(-1)^3 - 6(-1)^2 + 2$

$y = -2 - 6 + 2$ ✓

$y = -6$

when $x = -1$, $y = -6$.

$y = 18x + c$

$-6 = -18 + c$ ✓

$c = 12$

$\therefore a = 2$

$b = 6$

$c = 12$.

Name : ______________________________ Marks: $\overline{}$ 31

Section 2 : Calculator and Resource Assumed. **Time Allowed : 35 minutes**

Note : Show working for full marks to be awarded.

1. [2,2= 4 marks]

A company produces **n** items of a certain product.
The cost function \$C is given by $C(n) = 1200 + 5n^{1/3}$
Each item sells for \$52.

Find

a) An expression for the marginal profit $P'(n)$

$$C(n) = 1200 + 5n^{1/3}$$

$$R(n) = 52n$$

$$P(n) = 52n - 1200 - 5n^{1/3} \checkmark$$

$$P'(n) = 52 - \frac{5}{3}n^{-2/3}$$

$$P'(n) = 52 - \frac{5}{3n^{2/3}} \checkmark$$

b) A value for $P'(64)$ and comment on its meaning.

$$P'(64) = 52 - \frac{5}{3(64^{2/3})}$$

$$= \underline{51.90} \checkmark$$

Profit of $\underline{65^{th}}$ Unit is \$ $\underline{51.90}$. ✓

must have both to get mark.

2. [2,4,4 = 10 marks]

a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The **depth 'y'** of fluid in the tank **'t' hours after** the valve is opened is given by **$y = 6(1 - \frac{t}{12})^2$ metres.**

i) Show ,with full working out , the rate $\frac{dy}{dt}$ m/hour at which the tank is draining at time t is $\frac{t}{12} - 1$

$y = 6(1 - \frac{t}{12})^2$

$\frac{dy}{dt} = 12(1 - \frac{t}{12})^1(-\frac{1}{12}) = -(1 - \frac{t}{12})$ ✓ ✓

$= \frac{t}{12} - 1$

ii) **a) When** is the fluid in the tank falling **fastest** and **slowest** ?

Slowest : when $t = 12h$ ✓

Fastest : when $t = 0\ h$ ✓

b) **What** are the values of $\frac{dy}{dt}$ at these times?

Slowest: $\frac{dy}{dt} = 0$.

$\frac{t}{12} - 1 = 0$

$t = 12\ h$.

Fastest

$t = 0$, $\frac{dy}{dt} = \frac{0}{12} - 1$

$\frac{dy}{dt} = -1\ m/h$

Units not needed.

$\therefore$ Slowest: $\frac{dy}{dt} = 0\ m/h$ ✓ Fastest: $\frac{dy}{dt} = -1\ m/h$. ✓

b) If the volume of a cylinder is given by $V = 2\pi r^3$, find the approximate percentage change in V when r changes by $\frac{1}{2}$%.

$\frac{\delta r}{r} = \frac{1}{2}\% = 0.005$

$V = 2\pi r^3$

$\frac{dV}{dr} = 6\pi r^2$ ✓

$\frac{\delta V}{\delta r} \approx \frac{dv}{dr}$

$\delta V \approx \frac{dv}{dr} \times \delta r$.

$\delta V \approx 6\pi r^2 \times \delta r$.

$\frac{\delta V}{V} \approx \frac{6\pi r^2 \delta r}{2\pi r^3}$ ✓

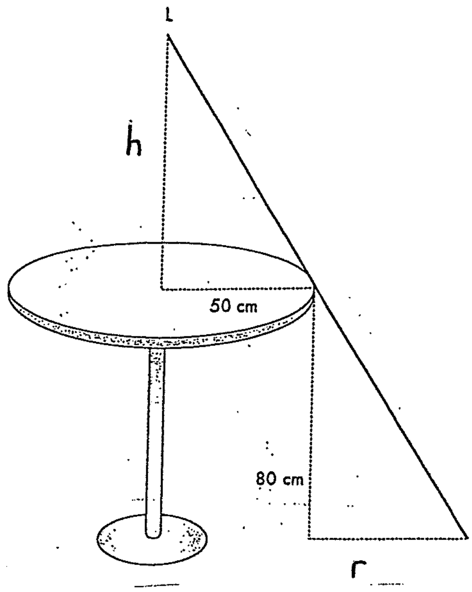
$\approx 3\frac{\delta r}{r}$ ✓

$\approx 3 \times 0.005$

≈ 0.015

$= 1.5\%$ ✓

3. [1,3 = 4 marks]



A table has a radius of 50 cm and a height of 80 cm.

A light (L) is lowered vertically downwards from a point above the centre of the table at a constant rate of 0.2 cm per second.

When the light is h cm above the table it casts a shadow that extends r cm from the edge of the table.

a) Show that r = $\frac{4000}{h}$

As triangles are similar, corresponding sides are in proportion.

$$\frac{r}{50} = \frac{80}{h}. \checkmark$$

$$\therefore r = \frac{4000}{h}$$

b) Find the rate at which r is changing when h = 60

$$\frac{dr}{dt} = \frac{dr}{dh} \cdot \frac{dh}{dt}$$

$$= \boxed{\frac{-4000}{h^2}} \times \frac{dh}{dt} \checkmark$$

$$\frac{dh}{dt} = -0,2 \checkmark$$

When $h = 60$

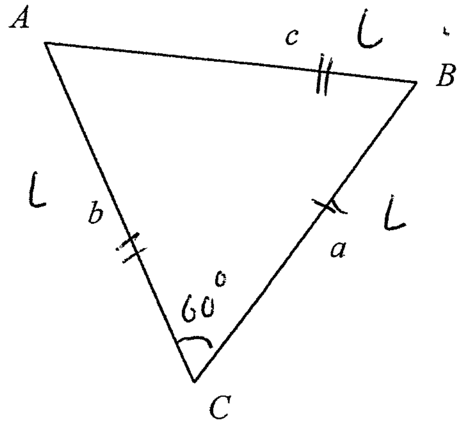
$$\frac{dr}{dt} = \frac{-4000}{60^2} \times (-0,2)$$

$$= \frac{2}{9} \text{ or } 0.\dot{2} \text{ cm/sec} \checkmark$$

$\therefore$ Radius is increasing at $2/9$ cm/sec.

4. [5 marks]

The area of a triangle can be found by the formula : Area $= \frac{ab \text{ Sin } C}{2}$



Using the **incremental formula**, determine the approximate change in area (to 3 decimal places) of **an equilateral triangle** with each side of 10 cm, when each side increases by 0.1cm.

[Hint : Use exact value for 60^0]

Δ ABC is an equilateral Δ

Let $a = b = c = l$ cm.

$\angle C = 60° = \frac{\sqrt{3}}{2}$

$$\text{Area} = \frac{1}{2} ab \sin\theta$$

$$= \frac{1}{2}(L)(L)\sin\frac{\pi}{3}$$

$$= \frac{1}{2} \cdot L^2 \cdot \frac{\sqrt{3}}{2}$$

$$A = \frac{\sqrt{3}}{4} L^2$$

$$\frac{dA}{dL} = \frac{2L \cdot \sqrt{3}}{4}$$

$L = 10$ cm

$\delta L = 0.1$ cm.

$$\frac{\delta A}{\delta L} \approx \frac{dA}{dL} \approx 2L \cdot \frac{\sqrt{3}}{4}.$$

$$\delta A \simeq 2L\frac{\sqrt{3}}{4} \cdot \delta L$$

$$\delta A \simeq 2(10) \cdot \frac{\sqrt{3}}{4} \cdot (0.1)$$

$$\delta A \simeq 0.8660254038$$

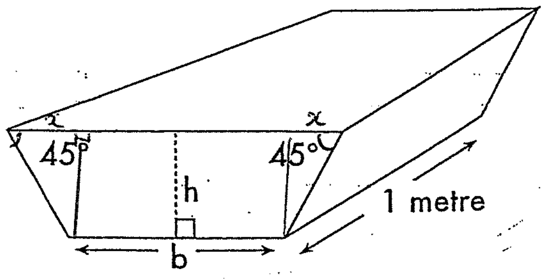
$$\delta A \simeq 0.866 \text{ sq cm } (3 \text{ dp})$$

$\therefore$ Approximate Change in area of 0.866 sq cm.

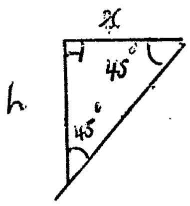
5. [3,2,3 = 8 marks]

An animal drinking trough is constructed from stainless steel in the shape of a trapezoidal prism, with height 'h' metres and length of 1 metre.

The cross section of the prism is an isosceles trapezium with acute angle of 45^0, base 'b' metres and area of 60 m^2.



a) Show that $b = \frac{60}{h} - h$



$\tan 45^\circ = \frac{h}{x}$

$b = x \tan 45^0$

$h = x(1)$

$h = x$ ✓

$h = x$ (Isos Δ).

$\text{Area} = \frac{1}{2}(b + b + 2h) \times h = 60$ ✓

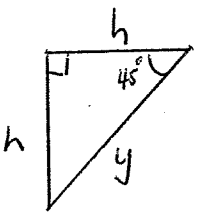
$\frac{2b+2h}{2} \times h = 60$

$\frac{\cancel{2}(b+h)h}{\cancel{2}} = 60$

$b + h = \frac{60}{h}$ ✓

$b = \frac{60}{h} - h$

b) Show that the surface area 'A' in m^2 is : $A = \frac{60}{h} - h + 2h\sqrt{2} + 120$



$y^2 = h^2 + h^2$

$y^2 = 2h^2$

$y = \sqrt{2}\,h$ ✓

$\text{Area} = A = 2 \times 60 + b(1) + 2\sqrt{2}\,h \times 1$

$A = 120 + b + 2\sqrt{2}\,h$

$b = \frac{60}{h} - h$

$A = 120 + \frac{60}{h} - h + 2\sqrt{2}\,h$ ✓

c) Find the depth of the drinking trough to the nearest mm, if the amount of stainless steel is to be kept to a minimum. Justify your answer by using Calculus techniques.

$$A = \frac{60}{h} - h + 2h\sqrt{2} + 120$$

$$A = 60h^{-1} - h + 2h\sqrt{2} + 120.$$

$$\frac{dA}{dh} = \underline{-60h^{-2} - 1 + 2\sqrt{2}} \checkmark$$

For minimum value $\frac{dA}{dh} = 0$.

$$-60h^{-2} - 1 + 2\sqrt{2} = 0 \checkmark$$

$$h = \underline{5.7284456\,57} \approx 5.728\text{ m } (3dp).$$

$$\frac{d^2A}{dh^2} = 120h^{-3} = \frac{120}{h^3}$$

$$\left.\frac{d^2A}{dh^2}\right|_{h=5.728} > 0 \quad \therefore \text{Rel. Min} \quad \checkmark$$

$\therefore$ Depth $= \underline{5.728 \text{ m}}$.

End of Test